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THEORETICAL ANALYSES OF VAPOUR CONDENSATION ON LAMINAR LIQUID JETS

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Abstract—The heating of a laminar liquid jet by the condensation of its saturated vapour is analysed. Solutions for heat transfer in circular and flat ducts held at constant wall temperature, are adapted to the problems of vapour condensation on a cylindrical jet and on a sheet of uniform thickness respectively. The solutions are extended to the case of constant external temperature with constant external resistance on the jet surface. A solution is also derived for the condensation heating of a fan spray sheet which has been the subject of an experimental investigation, reported separately [2].

Simplified equations are derived giving the local Nusselt number and the average temperature for jets with zero surface resistance, having a limiting low or a limiting high Graetz number. The simplified equations are shown to have a common form for all jet types considered.

Numerical values are presented for the local Nusselt number and the average temperature as a function of the Graetz number for jets with zero surface resistance.

NOMENCLATURE

A_n,	constants in the eigenvalue problems;
c,	specific heat;
C_1,	longitudinal co-ordinate in fan spray sheet derivation;
C_2,	transverse co-ordinate in fan spray sheet derivation;
$\bar{C}_2$,	boundary value of C_2;
D_H,	hydraulic diameter;
Gz,	Graetz number;
h,	heat-transfer coefficient at the jet surface;
h_x,	local overall heat-transfer coefficient;
K,	thermal conductivity;
l_1,	length co-ordinate in the longitudinal direction;
l_2,	length co-ordinate in the transverse direction;
M_1, M_2, M_3,	constants in limiting low Gz solutions;
n,	eigenvalue and summation indices;
N_1, N_2,	constants in limiting high Gz solutions;
Nu_x,	local Nusselt number;
p,	transformation of z;
q_x,	local heat flux;
Q,	volumetric flow rate;
r,	radius or radial co-ordinate;
R,	jet radius;
s,	sheet thickness;
T,	local jet temperature;
$\bar{T}$,	average jet temperature;
T_0,	jet inlet temperature;
T_s,	external temperature (the saturated vapour temperature);
T_w,	jet surface temperature;
u,	jet velocity in the flow direction;
x, y,	Cartesian co-ordinates;
z,	transformed Gz number.

Greek symbols

α,	thermal diffusivity;
θ,	non-dimensional local jet temperature;
$\bar{\theta}$,	non-dimensional average jet temperature;
λ_n,	eigenvalue;
ϕ,	angle subtended by fan spray sheet.

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INTRODUCTION

HEAT transfer in the direct contact condensation of steam on a water jet is of interest in the design of equipment such as contact condensers, contact feed water heaters and deaerators. This type of equipment, long used in industry, is becoming increasingly important in the development of economic water desalination units. A notable application of direct contact heat transfer between steam and water is found in the "vapor reheat flash process" for water desalination [1] which is currently being developed under the sponsorship of the U.S. Office of Saline Water.

Direct contact heating is a very efficient heat-transfer operation. Very high coefficients of heat transfer may be attained by condensing steam on a thin water sheet. Coefficients exceeding 200 000 kcal/h m^2 degC have been measured in the present investigation.

This paper forms the theoretical part of a mainly experimental study of steam condensation on laminar water sheets generated by fan spray nozzles. It describes theoretical solutions for the heating of laminar jets of various types by the condensation of saturated vapour. The analyses are simplified and made amenable to exact mathematical solution according to the following concept. The heating of a jet by the condensation of saturated vapour is considered to be analogous to the heating of a fluid in plug flow through a duct held at constant external temperature. The case of heat transfer with an external resistance at the wall is also examined.

Confirmation of the solution derived for the fan spray sheet with zero surface resistance, is included in the experimental part of this investigation which is reported separately [2].

The theory of jet heating by vapour condensation has been partly treated by Kutateladze [3] who examined freely falling cylindrical and plane jets respectively, assuming no surface resistance. An error was introduced in each of the analytical solutions, in the conversion of the basic differential equation into non-dimensional form. The variation in jet thickness, arising from gravitational acceleration, was ignored in the transformation of the thickness term. The solutions become valid if jet velocity is assumed constant and gravity terms are omitted.

Kutateladze formulated his solutions in a form applying also for turbulent heat transport, by adding the eddy diffusivity coefficient to the thermal diffusivity term. However, the eddy diffusivity coefficient for jets is unknown. It differs from that for pipe flow, since in jets the velocity profile is essentially uniform, while in pipes, the profile is non-uniform. For this reason, the experimental verification presented by Kutateladze for a turbulent cylindrical water jet heated by steam is inconclusive; it is based on an eddy diffusivity coefficient evaluated from pipe flow correlations.

GENERAL ASSUMPTIONS

The well known and extensively studied problem of filmwise condensation of a vapour, such as steam, on a cold solid surface [4] differs basically from the problem of vapour condensation on a liquid jet. In the former case a marked velocity gradient exists in the condensate film due to the shear stress at the solid boundary. Also, the heat of condensation is discarded through the cold solid surface so that the amount condensed is only limited by the extent of the surface.

In the latter case, the velocity profile is approximately uniform since the velocity gradient arising from the shear stress at a liquid–gas interface is usually negligibly small compared to that at a solid–liquid interface. Furthermore, all the heat of condensation is absorbed by the jet, raising its temperature. There is, therefore, a definite limit to the amount of vapour that may be condensed on a jet.

For a water jet initially at room temperature, the maximum amount of steam which can be condensed is less than 20 per cent of the initial flow. Mathematical analysis is considerably simplified by neglecting the flow variation along the jet, due to the condensing liquid. The problem becomes analogous to the heating of a fluid moving in plug flow through a duct held at a constant external temperature. Solutions are available for the case of the round duct and for the flat duct with no surface resistance [5]. These solutions will be briefly reviewed and compared to the solution derived for the fan spray sheet.

The case of external resistance at the surface, also treated below, is of considerable interest. Under practical conditions, a surface resistance may arise, for instance, from interfacial condensation resistance [6, 7] or from build-up of an inert gas layer on the condensation interface [8].

Simplifying assumptions common to the derivations presented below are as follows:

(a) The condensing vapour is saturated and hence, the jet moves in a medium of constant temperature T_s. The surface temperature of the jet, T_w, is evaluated assuming that the heat-transfer coefficient h, corresponding to the surface resistance, is constant along the jet. Obviously, with zero surface resistance, $T_w = T_s$.

(b) Jet velocity is assumed constant, neglecting the interfacial drag and the reaction on the jet due to vapour initially at rest condensing on a moving liquid.

(c) The usually small variations of physical properties (c, ρ, K) with temperature are neglected.

(d) Heat conduction in the flow direction is small compared to conduction in the transverse direction, and as usual, can be ignored.

The assumptions made, other than those calling for effects of the condensate sheath to be ignored, may be considered to be reasonably representative of practical conditions. The discrepancy arising from neglecting condensate effects appears to be small in the light of the experimental verification described elsewhere [2].

CYLINDRICAL JET

Using the symbols of Fig. 1, the well-known heat conduction equation for the cylindrical jet is as follows [5]:

$$u \frac{\partial T}{\partial x} = \alpha \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right). \tag{1}$$

Product solution of equation (1) gives the following expression for the temperature distribution:

$$\theta = \frac{T_s - T}{T_s - T_0} = \sum_{n=1}^{n=\infty} A_n \exp\left(\frac{-\lambda_n^2 \alpha x}{u R^2}\right) . J_0 \left(\lambda_n \frac{r}{R}\right) \tag{2}$$

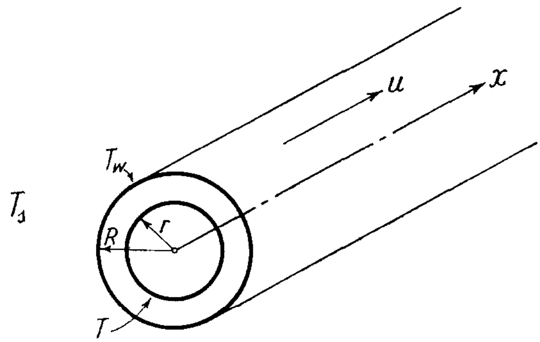

FIG. 1. Cylindrical jet nomenclature.

where J_0 is the zero order Bessel function of the first kind. Using the initial condition, $T = T_0$ at $x = 0$, the constants A_n are given by:

$$A_n = \frac{2 J_1(\lambda_n)}{\lambda_n \{[J_0(\lambda_n)]^2 + [J_1(\lambda_n)]^2\}} \tag{3}$$

where J_1 is the first order Bessel function of the first kind. Integration of equation (2) yields the following expression for the average temperature:

$$\bar{\theta} = \frac{T_s - \bar{T}}{T_s - T_0} = \sum_{n=1}^{n=\infty} \frac{2A_n J_1(\lambda_n)}{\lambda_n} \exp\left(\frac{-\lambda_n^2 \alpha x}{u R^2}\right). \tag{4}$$

The local heat-transfer coefficient is defined by:

$$q_x = 2\pi R \, dx \, h_x . (T_s - \bar{T}) = \rho u \pi R^2 c \, dx \, (\partial \bar{T}/\partial x). \tag{5}$$

Using equations (4) and (5), the general expression for the local Nusselt number is:

$$Nu_x = \frac{h_x D}{K} = \frac{\sum_{n=1}^{n=\infty} A_n \lambda_n J_1(\lambda_n) \exp(-\lambda_n^2 \alpha x/u R^2)}{\sum_{n=1}^{n=\infty} [A_n J_1(\lambda_n)/\lambda_n] \exp(-\lambda_n^2 \alpha x/u R^2)}. \tag{6}$$

Zero surface resistance

With no surface resistance, the boundary condition is $T = T_s$ at $r = R$. Equation (2) shows that the eigenvalues are roots of the J_0 function:

$$J_0(\lambda_n) = 0. \tag{7}$$

Hence, equations (4) and (6) simplify into:

$$\bar{\theta} = \sum_{n=1}^{n=\infty} \frac{4}{\lambda_n^2} \exp\left(-4\lambda_n^2/Gz\right) \tag{8}$$

$$Nu_x = \frac{\sum_{n=1}^{n=\infty} \exp\left(-4\lambda_n^2/Gz\right)}{\sum_{n=1}^{n=\infty} (1/\lambda_n^2) \exp\left(-\lambda_n^2/Gz\right)} \tag{9}$$

where the distance x is expressed non-dimensionally through a Graetz number based on the hydraulic diameter:

$$Gz = u\, D^2/\alpha x. \tag{10}$$

Approximations for limiting Graetz numbers

For small Graetz numbers, each of the series in equations (8) and (9) converges rapidly. Taking the first term of the series, the following well-known [5] approximations are obtained:

$$\bar{\theta} = 0{\cdot}6915 \exp\left(-23{\cdot}136/Gz\right) \tag{11}$$

$$Nu_x = 5{\cdot}784. \tag{12}$$

For large Graetz numbers, the following approximate relationships hold (see Appendix A):

$$\bar{\theta} = 1 - \frac{8}{\sqrt{(\pi\, Gz)}} \tag{13}$$

$$Nu_x = \frac{\sqrt{(Gz/\pi)}}{1 - 8/\sqrt{(\pi\, Gz)}} \approx \sqrt{\left(\frac{Gz}{\pi}\right)}. \tag{14}$$

Finite surface resistance

Here, the boundary condition at the surface is:

$$h\,(T_s - T_w) = K\left(\frac{\partial T}{\partial r}\right) \quad \text{at } r = R. \tag{15}$$

Using equation (2), it is found that the eigenvalues for this case are given by:

$$\frac{\lambda_n\, J_1(\lambda_n)}{J_0(\lambda_n)} = \frac{hD}{2K}. \tag{16}$$

The general expressions for A_n, $\bar{\theta}$ and Nu_x remain unaltered. The Nusselt number calculated with equation (16) is, by the definition, an overall heat-transfer coefficient combining the internal conduction resistance of the liquid and the external surface resistance.

For the limiting case of small Graetz numbers, a simple relationship for Nu_x can be obtained, by taking only the first term of the rapidly converging series. Equation (6) reduces to:

$$Nu_x = \lambda_1^2. \tag{17}$$

Hence, from equation (16):

$$\frac{\sqrt{(Nu_x)}\,.\,J_1(\sqrt{Nu_x})}{J_0(\sqrt{Nu_x})} = \frac{hD}{2K}. \tag{18}$$

PLANE SHEET OF UNIFORM THICKNESS

In an ideal sheet of uniform thickness s having an infinite width (Fig. 2), the heat conduction equation takes the form [5]:

$$\frac{\partial T}{\partial x} = \frac{\alpha}{u} \frac{\partial^2 T}{\partial y^2}. \tag{19}$$

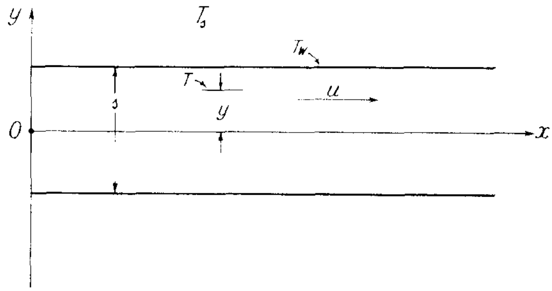

FIG. 2. Nomenclature for plane sheet of uniform thickness.

Product solution of this equation gives the temperature distribution as:

$$\theta = \frac{T_s - T}{T_s - T_0} =$$

$$\sum_{n=1}^{n=\infty} A_n \exp\left(\frac{-4\lambda_n^2\, \alpha\, x}{u\, s^2}\right).\, \cos\,(2\lambda_n\, y/s). \tag{20}$$

From the initial condition, $T = T_0$ at $x = 0$, the constants A_n are given by:

$$A_n = \frac{2 \sin \lambda_n}{\lambda_n + \sin \lambda_n \cos \lambda_n}. \tag{21}$$

The average temperature of a cross section, obtained by integration of equation (20), is:

$$\bar{\theta} = \frac{T_s - \bar{T}}{T_s - T_0} = \int_0^1 \theta \, d\left(\frac{2y}{s}\right) = \sum_{n=1}^{n=\infty} \frac{A_n \sin \lambda_n}{\lambda_n} \exp\left(\frac{-4\lambda_n^2 \alpha x}{u s^2}\right). \quad (22)$$

Taking into account the heat-transfer area on both sides of the plane, the local heat-transfer coefficient h_x is defined by:

$$q_x = 2\,dx \,.\, h_x \,.\, (T_s - \bar{T}) = \frac{\partial}{\partial x}(\rho u s c \bar{T})\, dx \quad (23)$$

where q_x is the local heat flux per unit breadth. Using equation (22), the general expression of the local Nusselt number, based on the hydraulic diameter of the infinitely wide sheet (2s), is found to be:

$$Nu_x = \frac{h_x \,.\, 2s}{K} = \frac{\sum_{n=1}^{n=\infty} 4\lambda_n^2 \frac{A_n \sin \lambda_n}{\lambda_n} \exp\left(\frac{-4\lambda_n^2 \alpha x}{u s^2}\right)}{\sum_{n=1}^{n=\infty} \frac{A_n \sin \lambda_n}{\lambda_n} \exp\left(\frac{-4\lambda_n^2 \alpha x}{u s^2}\right)}. \quad (24)$$

Zero surface resistance

In this case, the boundary condition at the surface is $T = T_s$ at $y = s/2$. From equation (20), the eigenvalues are:

$$\lambda_n = \frac{\pi}{2}(2n - 1). \quad (25)$$

Substituting for the values of the constants, and expressing the distance x by a Graetz number based on the hydraulic diameter of sheet, the following expressions are obtained:

$$\bar{\theta} = \frac{8}{\pi^2} \sum_{n=1}^{n=\infty} \frac{1}{(2n-1)^2} \exp\left[-4\pi^2 (2n-1)^2/Gz\right] \quad (26)$$

$$Nu_x = \pi^2 \frac{\sum_{n=1}^{n=\infty} \exp\left[-4\pi^2 (2n-1)^2/Gz\right]}{\sum_{n=1}^{n=\infty} \frac{1}{(2n-1)^2}\left[-4\pi^2 (2n-1)^2/Gz\right]} \quad (27)$$

where,

$$Gz = \frac{(2s)^2 u}{\alpha x} \quad (28)$$

Approximations for limiting Graetz numbers

For small Graetz numbers, the following simplified equations, based on the first term of the rapidly converging series, may be used:

$$\bar{\theta} = \frac{8}{\pi^2} \exp(-4\pi^2/Gz) \quad (29)$$

$$Nu_x = \pi^2. \quad (30)$$

For very large Graetz numbers, expansion into MacLaurin's series, as shown in Appendix A, gives the following approximate relationships:

$$\bar{\theta} = 1 - \frac{8}{\sqrt{(\pi\, Gz)}} \quad (31)$$

$$Nu_x = \frac{\sqrt{(Gz/\pi)}}{1 - [8/\sqrt{(\pi\, Gz)}]} \approx \sqrt{\left(\frac{Gz}{\pi}\right)}. \quad (32)$$

Finite surface resistance

The boundary condition at the surface, viz.:

$$h(T_s - T_w) = K \frac{\partial T}{\partial y} \qquad \text{at } y = s/2 \quad (33)$$

is used to determine the eigenvalues. From equation (20):

$$\lambda_n \tan \lambda_n = \frac{h s}{2K}. \quad (34)$$

In this case, the Nusselt number calculated from equations (21), (24) and (34) is an overall heat-transfer coefficient combining the conduction resistance and the surface resistance

A simple relationship can be derived for the limiting case of small Graetz numbers. Taking the first term of the rapidly converging series in equation (24), it is found that:

$$Nu_x = 4\lambda_1^2. \quad (35)$$

Hence, from equation (34)

$$\sqrt{(Nu_x)} \tan [\sqrt{(Nu_x)}/2] = \frac{h\,s}{K} \tag{36}$$

FAN SPRAY SHEET

The plane sheet of uniform thickness treated above, represents an ideal flow model which cannot be physically realized. If a liquid flows as a flat sheet, its thickness continually decreases in the flow direction.

The fan spray sheet [9] may be described as a sector of an attenuating disc subtending an angle of ϕ radians. The streamlines flow radially from the disc origin. Neglecting the small transverse velocity component involved in the spreading of the sheet, the velocity can be shown to be uniform throughout the sheet [9]. The continuity equation

$$Q = \phi x\, s\, u \tag{37}$$

shows that

$$x\, s = \text{constant} \tag{38}$$

i.e. sheet thickness s varies inversely with the distance x from the origin. Since conditions near the origin are indeterminate, it is assumed that the fan spray sheet of initial temperature T_0 is exposed to saturated vapour starting from a very short distance x_0 from the origin. It is further assumed that the sheet is thin compared to its radial extent so that $s \ll x$ at all points of the heated sheet. These assumptions are reasonably representative of practical conditions existing in a thin laminar sheet generated by a fan spray nozzle.

Analytical solution is obtained by using transformed co-ordinates, C_1 in the longitudinal direction and C_2 in the transverse direction (Fig. 3). Consistent with equation (38), liquid is assumed to flow in the longitudinal direction between surfaces of constant C_2 defined by:

$$C_2^2 = 2x\, y. \tag{39}$$

At the boundary,

$$\bar{C}_2^2 = 2x \frac{s}{2} = x\, s. \tag{40}$$

The co-ordinate C_1, normal to C_2, is determined by:

$$C_1^2 = x^2 - y^2. \tag{41}$$

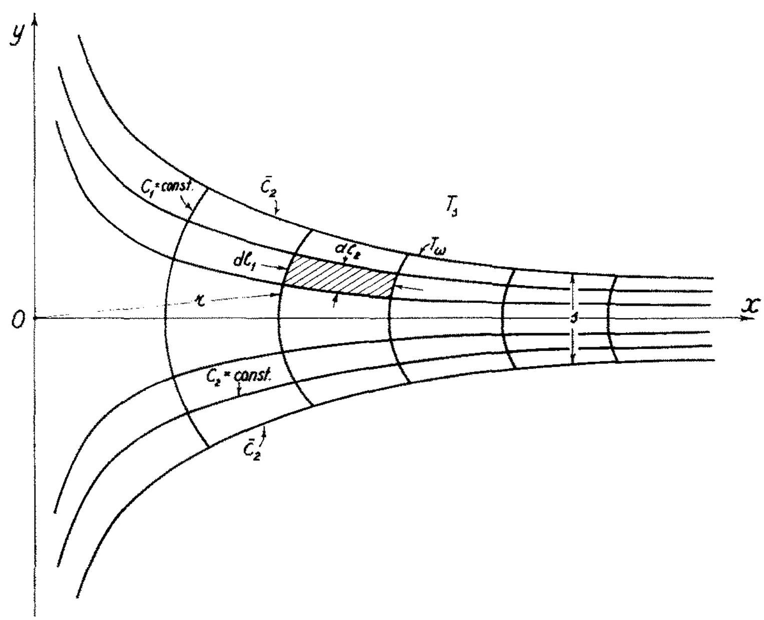

FIG. 3. Fan spray sheet nomenclature.

A heat balance over an element of length dl_1 in the longitudinal direction and width dl_2 in the transverse direction (Fig. 3) gives:

$$\frac{\partial}{\partial l_1}(\phi x \, dl_2 \, u \, \rho \, c \, T) dl_1 = \frac{\partial}{\partial l_2}\left(K \phi x \, dl_1 \frac{\partial T}{\partial l_2}\right) dl_2. \quad (42)$$

It is shown in Appendix B that:

$$\left(\frac{\partial l}{\partial C_1}\right)_{C_2} \equiv \frac{\partial l_1}{\partial C_1} = \frac{C_1}{r} \quad (43)$$

$$\left(\frac{\partial l}{\partial C_2}\right)_{C_1} \equiv \frac{\partial l_2}{\partial C_2} = \frac{C_2}{r}. \quad (44)$$

Using these relationships, equation (42) becomes:

$$u \frac{\partial}{\partial C_1}\left(\frac{xT}{r}\right) = \frac{\alpha}{C_2} \frac{\partial}{\partial C_2}\left(x \frac{C_1}{C_2} \frac{\partial T}{\partial C_2}\right). \quad (45)$$

Since $y \leqslant (s/2) \ll x$, it follows that $C_2 \ll C_1$ and hence:

$$C_1 \approx x \approx r. \quad (46)$$

Equation (45) thus simplifies into:

$$\frac{\partial T}{\partial x^3} = \frac{4\alpha}{3u}\left(\frac{\partial}{\partial C_2^2}\right)^2 T. \quad (47)$$

Since this expression is of the same form as equation (19) for a plane sheet of uniform thickness, its solution is easily derived by inserting in equation (20) appropriate substitutions. It is found that:

$$\theta = \frac{T_s - T}{T_s - T_0} =$$

$$\sum_{n=1}^{n=\infty} A_n \exp\left\{\frac{-4\lambda_n^2 \, \alpha (x^3 - x_0^3)}{3u \, \bar{C}_2^4}\right\} . \cos\left(\lambda_n \frac{C_2^2}{\bar{C}_2^2}\right). \quad (48)$$

The initial condition is: $T = T_0$ at $C_1 = x_0$ and with zero surface resistance the boundary condition is: $T = T_s$ at $C_2 = \bar{C}_2$. Using these boundary conditions, the eigenvalues are exactly those given by equation (25).

The average temperature of a transverse section defined by the surface C_1^2 = constant, is:

$$\bar{T} = \frac{\int_0^{l_2} T \, dl_2}{\int_0^{l_2} dl_2} = \int_0^1 T\left(\frac{C_2}{\bar{C}_2}\right) d\left(\frac{C_2}{\bar{C}_2}\right). \quad (49)$$

Integrating equation (48) and substituting for the constants, the average temperature is:

$$\bar{\theta} = \frac{8}{\pi^2} \sum_{n=1}^{n=\infty} \frac{1}{(2n-1)^2} \exp\left[-4\pi^2 (2n-1)^2 / 3\, Gz\right] \quad (50)$$

where the Graetz number, based on the hydraulic diameter of the sheet $2s$, is defined by:

$$Gz = \frac{4\bar{C}_2^4 u}{\alpha x^3 \{1 - (x_0/x^3)\}}. \quad (51)$$

Since the assumptions made restrict the solution to values of $x \gg x_0$, equation (51) simplifies into:

$$Gz = \frac{(2s)^2 \, u}{\alpha \, x}. \quad (52)$$

The local heat-transfer coefficient, taking into account the heating on both sides of the sheet, is given by:

$$q_x = (2\phi \, C_1 \, dl_1) . h_x . (T_s - \bar{T})$$

$$= \frac{\partial}{\partial l_1} (\rho \, u \, \phi \, \bar{C}_2^2 \, c \, \bar{T}) \, dl_1. \quad (53)$$

Hence using equations (43) and (46),

$$2h_x \, \bar{\theta} = -\rho \, u \, c \, s \frac{\partial \bar{\theta}}{\partial x}. \quad (54)$$

Differentiating equation (50) and rearranging, it is found that

$$Nu_x = \frac{h_x . 2s}{K} =$$

$$\pi^2 \frac{\sum_{n=1}^{n=\infty} \exp\left[-4\pi^2 (2n-1)^2/3\, Gz\right]}{\sum_{n=1}^{n=\infty} \frac{1}{(2n-1)^2} \exp\left[-4\pi^2 (2n-1)^2/3\, Gz\right]}. \quad (55)$$

Comparison of equations (50) and (55) with the corresponding equations (26) and (27) for the sheet of uniform thickness shows that the expressions are identical except for the factor 3 preceding Gz in the case of the fan spray sheet. Hence the following approximate relationships may be immediately written by analogy to the case of a sheet of uniform thickness. For small Graetz numbers,

$$\bar{\theta} = \frac{8}{\pi^2} \exp(-4\pi^2/3\ Gz) \tag{56}$$

$$Nu_x = \pi^2. \tag{57}$$

For large Graetz numbers,

$$\bar{\theta} = 1 - \frac{8}{\sqrt{(3\pi\ Gz)}} \tag{58}$$

$$Nu_x = \frac{\sqrt{(3\ Gz/\pi)}}{1 - [8/\sqrt{(3\pi\ Gz)}]} \approx \sqrt{(3\ Gz/\pi)}. \tag{59}$$

The case of constant surface resistance is difficult to solve. Equation (34) does not hold since the eigenvalues become variables depending on s. The eigenvalues can be made constant by assuming that h varies inversely with s, but a solution based on this assumption seems trivial.

CONCLUSIONS

Some general conclusions can be noted by comparing the solutions of the different jets for zero surface resistance. The Graetz number has been defined on a common basis, using the hydraulic diameter of the jet D_H. In each case, the average temperature and the Nusselt number solely depend on the Graetz number. For small Graetz numbers, the local Nusselt number tends to a constant value independent of the Graetz number.

Tables 1 and 2 give accurate values of $\bar{\theta}$ and Nu_x as a function of Gz, calculated by computer with an error of less than 10^{-6}. The functions are plotted in Figs. 4 and 5. Values of $\bar{\theta}$ and Nu_x can be calculated with less than 1 per cent error over most of the Graetz number range, using the simple relationships derived for limiting values of Gz. The magnitude of each limiting Gz giving the maximum error of 1 per cent in $\bar{\theta}$ or Nu_x has been determined (Table 3 and 4) by comparing computer results with values obtained from the approximate relationships.

For small Graetz numbers, the approximate relationships have the following common form:

$$-\log \bar{\theta} = M_1 + (M_2/Gz) \tag{60}$$

$$Nu_x = M_3. \tag{61}$$

Table 1. Values of $\bar{\theta}$ and Nu$_x$ *as a function of* Gz *for the cylindrical jet with zero surface resistance*

Gz	$\bar{\theta}$	Nu_x
400 000	0·99288	358·373
200 000	0·98994	253·865
100 000	0·98578	179·969
66 666	0·98259	147·234
50 000	0·97991	127·720
40 000	0·97754	114·403
20 000	0·96030	81·363
10 000	0·95528	58·007
6 666	0·94532	47·664
5 000	0·93698	41·501
4 000	0·92965	37·297
2 000	0·90110	26·876
1 000	0·86133	19·531
666	0·83130	16·292
500	0·80607	14·372
400	0·78454	13·068
200	0·70144	9·884
100	0·58802	7·744
66·6	0·51016	6·886
50	0·44709	6·437
40	0·39419	6·179
26·6	0·29187	5·898
20	0·21782	5·817
16	0·16299	5·793
13·33	0·12203	5·786
11·43	0·09138	5·784
10	0·06844	5·783
8·88	0·05125	5·783
8	0·03838	5·783
4	0·00213	5·783
2	0·000066	5·783

Values of the constants M_1, M_2 and M_3 for the different jets are summarized in Table 3.

Similarly, for large Graetz numbers, the expressions have the following common form:

$$\bar{\theta} = 1 - (N_1/\sqrt{Gz}) \tag{62}$$

$$Nu_x = \frac{N_2\sqrt{Gz}}{1 - (N_1/\sqrt{Gz})}. \tag{63}$$

Values of the constants N_1 and N_2 are given in Table 4.

It is of interest to note that the solution for the uniformly thick sheet shows similarity to that for the cylindrical jet on the one hand and to that for the fan spray sheet on the other hand. At large Graetz numbers (the entry region) the expressions for $\bar{\theta}$ and Nu_x for the uniformly thick sheet and for the cylindrical

Table 2. Values of $\bar{\theta}$ and Nu_x *as a function of* Gz *for the uniformly thick sheet* (Gz_1) *and for the fan spray sheet* (Gz_2) *with zero surface resistance*

Gz_1	Gz_2	$\bar{\theta}$	Nu_x	Gz_1	Gz_2	$\bar{\theta}$	Nu_x
3 944 000	1 315 000	0·99867	1 123·55	985·9	328·6	0·85710	20·698
1 972 000	657 300	0·99773	795·22	657·3	219·1	0·82482	17·562
985 900	328 600	0·99640	563·06	492·9	164·3	0·79757	15·729
657 300	219 100	0·99538	460·21	394·4	131·5	0·77357	14·505
492 900	164 300	0·99452	398·90	262·9	87·64	0·72246	12·631
394 400	131 500	0·99375	357·06	197·2	65·73	0·67939	11·678
197 200	65 730	0·99077	253·23	157·8	52·584	0·64143	11·062
98 590	32 860	0·98657	179·83	131·5	43·822	0·60713	10·664
65 730	21 910	0·98334	147·31	112·7	37·560	0·57561	10·401
49 290	16 430	0·98061	127·93	98·595	32·865	0·54632	10·226
39 440	13 150	0·97821	114·71	87·639	29·213	0·51890	10·108
19 720	6 573	0·96882	81·898	78·876	26·292	0·49311	10·030
9 859	3 286	0·95547	58·717	39·433	13·146	0·29849	9·873
6 573	2 191	0·94525	48·451	19·719	6·573	0·10982	9·870
4 929	1 643	0·93663	42·354	13·146	4·382	0·04039	9·870
3 944	1 315	0·92904	38·192	9·860	3·287	0·01486	9·870
1 972	657·3	0·89926	27·900	7·888	2·629	0·00547	9·870

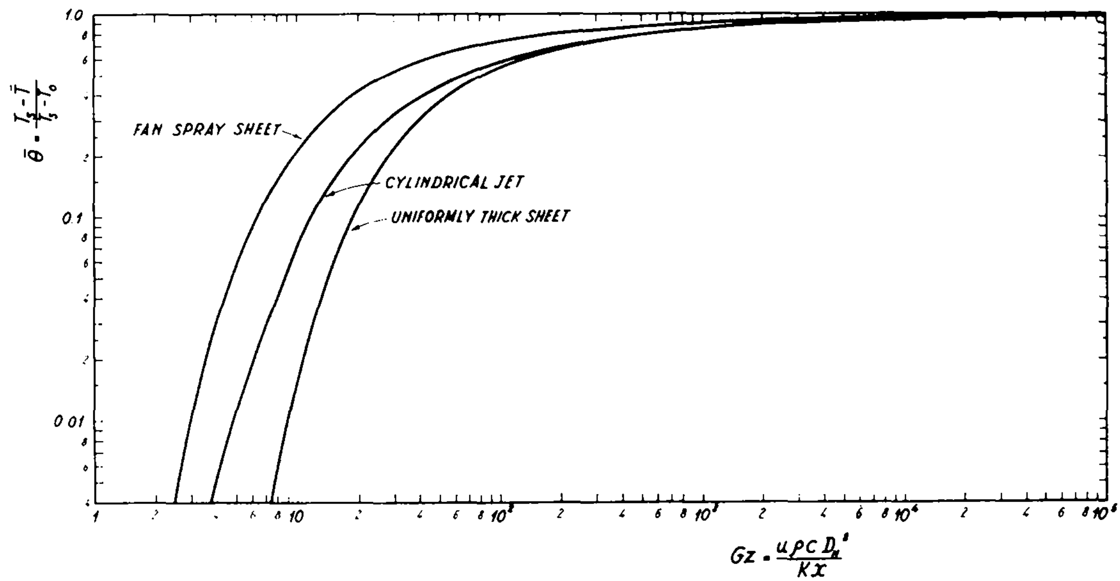

FIG. 4. Average temperature as a function of the Graetz number for jets of various configurations (zero surface resistance).

jet are identical (Table 4). At small Graetz numbers (the region of fully developed profile) Nu_x has the same constant value for the uniformly thick sheet and for the fan spray sheet (Table 3).

When considering the local heat-transfer coefficient h_x, an important difference between the uniformly thick sheet and the fan spray sheet should be noted. Referring to equations (32) and (30), it is seen that for the sheet of uniform thickness, the local heat-transfer

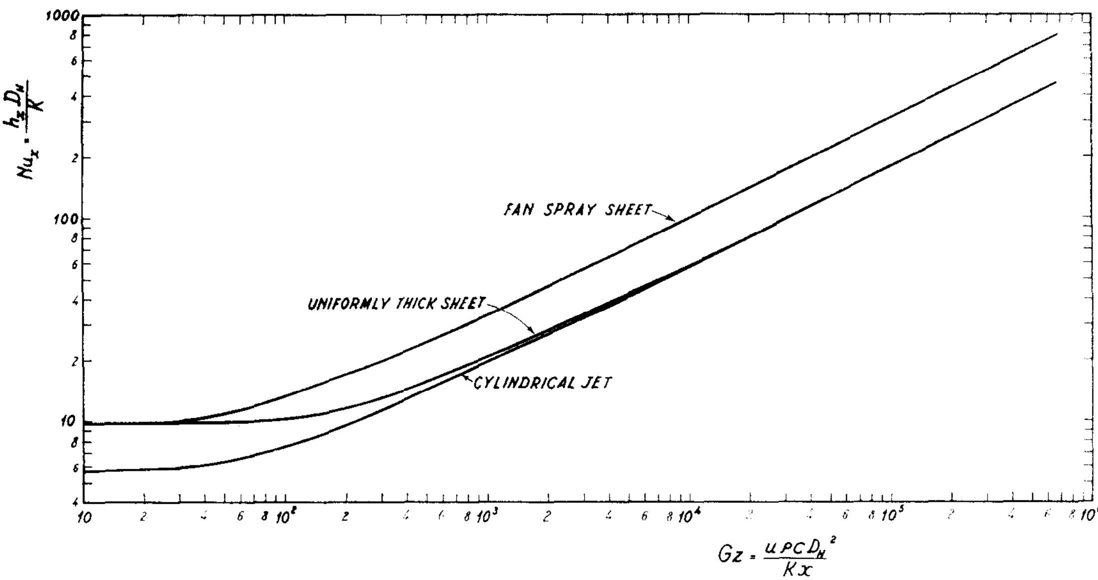

FIG. 5. Local Nusselt number as a function of the Graetz number for jets of various configurations (zero surface resistance).

Table 3. Values of constants in approximate equations for limiting low Graetz numbers and upper values of Gz *for which the equations apply with* 1 *per cent accuracy*

Shape of jet	M_1	M_2	M_3	Upper value of Gz for $\bar{\theta}$	Upper value of Gz for Nu_x
Cylindrical	0·160	9·08	5·784	70	25
Uniformly thick sheet	0·092	17·20	9·870	120	60
Fan spray sheet	0·092	5·75	9·870	40	20

Table 4. Values of constants in approximate equations for limiting high Graetz numbers and lower values of Gz *for which the equations apply with* 1 *per cent accuracy*

Shape of jet	N_1	N_2	Lower value of Gz for $\hat{\theta}$	Lower value of Gz for Nu_x
Cylindrical	4·5135	0·56419	500	10 000
Uniformly thick sheet	4·5135	0·56419	80	80
Fan spray sheet	2·6132	0·97999	26	26

coefficient h_x continuously diminishes as x is increased reaching a constant value in the region of practical saturation. In the fan spray sheet, the decrease in sheet thickness with increasing x reduces the effective thickness of the thermally resistant layer and tends to augment h_x as x is increased. Equations (38), (52), (57) and (59) show that for low values of x, corresponding to large Gz, h_x decreases according to:

$$h_x \propto \frac{1}{\sqrt{x}} \tag{64}$$

while in the region of large x, corresponding to small Gz, h_x increases according to:

$$h_x \propto x \tag{65}$$

Equations (64) and (65) can explain a striking difference between Weinberg's experimental results [10] for conical spray sheets and those obtained for fan spray sheets in the experimental part of this work [2]. Although in both forms of sheet the thickness variation with x is of a similar type, h_x was found to diminish with increasing x in the former investigation while an opposite trend was observed in the present investigation.

The variation of h_x with x will be seen to be in the right direction in each case if it is realized that Weinberg studied thick sheets of high Gz, while this work was confined to thin sheets of low Gz.

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APPENDIX A

Approximate Relationships for Large Graetz Numbers

At very large Graetz numbers, simplified equations may be derived, based on a two term expansion of the McLaurin's series. Let z be a variably inversely related to Gz (as defined below). For large Gz or small z, the following approximation holds:

$$\underset{z \to 0}{f(z)} = f(0) + zf'(0). \tag{A1}$$

Cylindrical jet

The following substitution will be used for convenience:

$$z = 2/\sqrt{Gz}. \tag{A2}$$

Equation (8) for $\bar{\theta}$ becomes,

$$\bar{\theta} = f(z) = \sum_{n=1}^{n=\infty} \frac{4}{\lambda_n^2} \exp(-\lambda_n^2 z^2). \tag{A3}$$

The initial condition, $\bar{\theta} = 0$ at $x = z = 0$, shows that $f(0) = 1$. Differentiating equation (A3):

$$f'(z) = \sum_{n=1}^{n=\infty} -8z \exp(-\lambda_n^2 z^2). \tag{A4}$$

The following approximation is justified for small values of z:

$$f'(z) \approx \int_{n \approx (1/4)}^{\infty} -8z \exp(-\lambda_n^2 z^2)\, dn. \tag{A5}$$

For the series [11] giving the roots of

$$J_0(\lambda_n) = 0 \tag{7}$$

the following approximation may be made when many terms are used:

$$\lambda_n \approx \pi \left(n - \frac{1}{4}\right). \tag{A6}$$

Hence

$$f'(z) = \int_0^{\infty} -\frac{8}{\pi} \exp(-p^2)\, dp = \frac{-4}{\sqrt{\pi}} = f'(0) \tag{A7}$$

where

$$p = \pi \left(n - \frac{1}{4}\right) z \tag{A8}$$

and

$$dp = \pi\, z\, dn. \tag{A9}$$

Substituting in equation (A1)

$$f(z) = \bar{\theta} = 1 - \frac{8}{\sqrt{(\pi\ Gz)}}. \tag{A10}$$

This approximation can be also applied to the denominator of Nu_x in equation (9). For the numerator, the following approximation holds for small values of z:

$$f(z) = \sum_{n=1}^{n=\infty} \exp(-\lambda_n^2 z^2)$$

$$\approx \frac{1}{z\pi} \int_0^\infty \exp(-p^2)\, dp = \frac{1}{2\sqrt{(\pi z)}}. \tag{A11}$$

Hence,

$$Nu_x = \frac{\sqrt{(Gz/\pi}}{1 - 8/\sqrt{(\pi\ Gz)}}. \tag{14}$$

Flat sheet of uniform thickness

The expression for $\bar{\theta}$ is:

$$\bar{\theta} = f(z) = \frac{8}{\pi^2} \sum_{n=1}^{n=\infty} \frac{1}{(2n-1)^2} \exp[-(2n-1)^2 z^2] \tag{A12}$$

and here,

$$z = \frac{2\pi}{\sqrt{Gz}}. \tag{A13}$$

The initial conditions give as before $f(0) = 1$, a result that can be checked from equation (26), using the well-known formula:

$$\sum_{n=1}^{n=\infty} \frac{1}{(2n-1)^2} = \frac{8}{\pi^2}. \tag{A14}$$

Differentiating equation (A12),

$$f'(z) = -\frac{16}{\pi^2} \sum_{n=1}^{n=\infty} z \exp[-(2n-1)^2 z^2]. \tag{A15}$$

As before, for small values of z,

$$f'(z) \approx \frac{8}{\pi^2} \int_0^\infty \exp(-p^2) \,.\, dp = \frac{4\sqrt{\pi}}{\pi^2} = f'(0) \tag{A16}$$

where

$$p = (2n-1)\, z$$

and

$$dp = 2z\, dn. \tag{A17}$$

Hence, substituting in equation (A12),

$$f(z) = \bar{\theta} = 1 - \frac{8}{\sqrt{(\pi\ Gz)}} \tag{31}$$

This expression can also be applied to evaluate the denominator of the Nu_x expression (equation 27). For small z, the numerator is given by:

$$f(z) = \sum_{n=1}^{n=\infty} \exp[-(2n-1)^2 z^2]$$

$$\approx \frac{1}{2z} \int_0^\infty \exp(-p^2)\, dp = \frac{\sqrt{\pi}}{4z}. \tag{A18}$$

Hence substituting in equation (27):

$$Nu_x = \frac{\sqrt{(Gz/\pi)}}{1 - [8/\sqrt{(\pi\ Gz)}]}. \tag{32}$$

Fan spray sheet

Substituting $3\,Gz$ for Gz in equations (31) and (32), the corresponding expressions for the fan spray sheet are immediately obtained.

APPENDIX B

Transformation of Co-ordinates in Fan Spray Sheet Derivation

The equations defining the C_1 and C_2 system of co-ordinates (Fig. 3) are:

$$C_2^2 = 2y\, x \tag{39}$$

$$C_1^2 = x^2 - y^2. \tag{40}$$

The derivative of interest $[(\partial l/\partial C_1)]_{C_2}$, is calculated from the following expression:

$$\left(\frac{\partial l}{\partial C_1}\right)_{C_2}^2 = \left(\frac{\partial x}{\partial C_1}\right)_{C_2}^2 + \left(\frac{\partial y}{\partial C_1}\right)_{C_2}^2. \tag{B1}$$

Partial differentiation of equations (39) and (40) with respect to C_1, at constant C_2, gives:

$$0 = 2y \left(\frac{\partial x}{\partial C_1}\right)_{C_2} + 2x \left(\frac{\partial y}{\partial C_1}\right)_{C_2} \tag{B2}$$

$$2C_1 = 2x \left(\frac{\partial x}{\partial C_1}\right)_{C_2} - 2y \left(\frac{\partial y}{\partial C_1}\right)_{C_2}. \tag{B3}$$

Solving simultaneously,

$$\left(\frac{\partial x}{\partial C_1}\right)_{C_2} = \frac{xC_1}{x^2 + y_2} \tag{B4}$$

$$\left(\frac{\partial y}{\partial C_1}\right)_{C_2} = \frac{-y\, C_1}{x^2 + y^2}. \tag{B5}$$

Substituting in equation (B1), noting that $x^2 + y^2 = r^2$, it is found that,

$$\left(\frac{\partial l}{\partial C_1}\right)_{C_2} \equiv \left(\frac{\partial l_1}{\partial C_1}\right)_{C_2} = \frac{C_1}{r}. \tag{43}$$

A similar derivation shows that,

$$\left(\frac{\partial l}{\partial C_2}\right)_{C_1} \equiv \left(\frac{\partial l_2}{\partial C_2}\right)_{C_1} = \frac{C_2}{r}. \tag{44}$$

Résumé—On analyse théoriquement l'échauffement d'un jet laminaire de liquide par la condensation de sa vapeur saturée. Des solutions pour le transport de chaleur dans des tuyaux circulaires et des tuyaux plats maintenus à une température pariétale constante, sont adaptées aux problèmes de la condensation de la vapeur respectivement sur un jet cylindrique et sur une lame d'épaisseur uniforme. Les solutions sont étendues au cas d'une température extérieure constante avec une résistance extérieure constante sur la surface du jet. On a aussi obtenu une solution pour l'échauffement par condensation d'un jet plan en éventail, qui a été le sujet d'une recherche expérimentale, rapportée séparément [2].

On a obtenu des équations simplifiées donnant le nombre de Nusselt local et la température moyenne pour des jets avec une résistance de surface nulle, dans les cas limites d'un nombre de Graetz faible ou élevé. On montre que les équations simplifiées ont une forme commune pour tous les types considérés de jets.

Des valeurs numériques sont présentées pour le nombre de Nusselt local et la température moyenne en fonction du nombre de Graetz pour des jets avec une résistance de surface nulle.

Zusammenfassung—Die Beheizung eines laminaren Flüssigkeitsstrahles durch kondensierenden Dampf wird analysiert. Lösungen für den Wärmeübergang in kreisförmigen und flachen Kanälen mit konstanter Wandtemperatur werden für die Probleme der Dampfkondensation an einem zylindrischen Strahl bzw. einer Platte gleichmässiger Dicke herangezogen. Die Lösungen werden auf den Fall konstanter Temperatur bei konstantem Wärmeübergangswiderstand an der Strahloberfläche ausgedehnt. Für die Beheizung eines ebenen Sprühfächers mit kondensierendem Dampf, die experimentell untersucht wurde, und worüber getrennt berichtet wird [2], ist eine Lösung abgeleitet.

Vereinfachte Gleichungen geben die örtliche Nusselt-Zahl und die mittlere Strahltemperatur bei einem Übergangswiderstand von Null, für eine obere und untere Grenze der Graetz-Zahl an. Die vereinfachten Gleichungen besitzen für alle betrachteten Strahltypen eine gemeinsame Form.

Numerische Werte sind für die örtliche Nusselt-Zahl und die Mitteltemperatur als Funktion der Graetz-Zahl für Strahlen mit dem Übergangswinkel Null angegeben.

Аннотация—Рассматривается нагрев ламинарной жидкой струи при конденсации ее насыщенного пара. Решения для теплообмена в круглых и плоских каналах при постоянной температуре стенки применяются соответственно к задачам о конденсации пара на цилиндрической струе и на струе в виде тонкой пластинки равномерной толщины. Дается обобщение этих решений для случаев постоянной внешней температуры при постоянном внешнем сопротивлении на поверхности струи. Найдено решение для нагрева веерной стури при конденсации, чему посвящено отдельное исследование (ссылка [2]).

Получены упрощенные уравнения для локального числа Нуссельта и средней температуры для струй с нулевым поверхностным сопротивлением при предельно низких или предельно высоких числах Гретца. Показано, что эти уравнения имеют общую форму для всех рассматриваемых типов струй.

Приводятся численные значения локального числа Нуссельта и средней температуры стенки в зависимости от числа Гретца для струй с нулевым поверхностным сопротивлением.